Ideals of nowhere dense sets in some topologies on integers

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Preliminaries

$\mathbb{N} = \{1, 2, 3, \ldots\}$ – the set of positive integers
$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ – the set of non-negative integers

For all $a, b \in \mathbb{N}$ the symbol $\{an + b\}$ stands for the infinite arithmetic progression with the initial term b and the difference a:

$$\{an + b\} = \{an + b \ : \ n \in \mathbb{N}_0\} = \{b,\ b + a,\ b + 2a,\ \ldots\}.$$

(a, b) - the greatest common divisor of a and b

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Three topologies

One can consider three topologies on $\mathbb{N}$:

- Furstenberg's topology $\mathcal{T}_F$ with the base
  $\mathcal{B}_F = \{\{an+b\} \ : \ b \leq a\}$;
  [normal, metrizable, zero-dimensional, totally disconnected]
- Golomb's topology $\mathcal{T}_G$ with the base
  $\mathcal{B}_G = \{\{an+b\} \ : \ (a,b)=1, \ b<a\}$;
  [Hausdorff but not regular, connected, not locally connected]
- Kirch's topology $\mathcal{T}_K$ with the base
  $\mathcal{B}_K = \{\{an+b\} \ : \ (a,b)=1, \ b<a, \ a - \text{square-free}\}$.
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Three ideals

An ideal on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements. We assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets.

Obviously, in any (decent) topology, the nowhere dense sets form an ideal. Let us then define three ideals on $\mathbb{N}$:

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Example (P. Szczuka)

$\mathbb{P} \in \mathcal{I}_F$, but $\mathbb{P}$ is dense in $\mathcal{T}_G$ and $\mathcal{T}_K$ (therefore it does not belong to $\mathcal{I}_G$ nor $\mathcal{I}_K$).

Example (M. Kwela, A. Nowik)

The set of even numbers $\{2n+2\}$ is in $\mathcal{I}_G$ and $\mathcal{I}_K$, but it belongs to the base of $\mathcal{T}_F$ (therefore $\{2n+2\} \notin \mathcal{I}_F$).

Theorem (M. Kwela)

$\mathcal{I}_K \subseteq \mathcal{I}_G$.

Theorem (M. Kwela, A. Nowik)

$\mathcal{I}_F$, $\mathcal{I}_G$ and $\mathcal{I}_K$ are $F_{\sigma\delta}$ but not F_σ ideals.

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Topological representations

We show that our new ideals (on countable set) may be connected to some σ-ideals in separable metrizable spaces. This connection has been introduced by M. Sabok and J. Zapletal.

Definition (M. Sabok, J. Zapletal)

Suppose that X is a separable metrizable space, $D \subseteq X$ – a dense countable set, and I – σ-ideal on X containing all singletons. Then

$$\mathcal{J}_I = \{A \subseteq D \ : \ cl(A) \in I\}$$

is an ideal on D.
Given an ideal $\mathcal{I}$ on $\mathbb{N}$ we say that $\mathcal{I}$ has a topological representation if there are I, D, X as above, for which $\mathcal{I}$ is isomorphic to $\mathcal{J}_I$ (i.e. there exists a bijection $f : \mathbb{N} \to D$ such that $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{J}_I$).
In such a case we say that $\mathcal{I}$ is represented on X by I.

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Topological representations

So far, only two examples of ideals with topological representations have been studied, namely:

$$\mathrm{NWD} = \{A \subseteq \mathbb{Q} \cap [0,1] \ : \ cl(A) \text{ is meager}\},$$

$$\mathrm{NULL} = \{A \subseteq \mathbb{Q} \cap [0,1] \ : \ cl(A) \text{ is of Lebesgue measure zero}\},$$

which are the ideals represented on $[0,1]$ by σ-ideals of meager sets and sets of Lebesgue measure zero, respectively.

Definition

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$.

(i) $\mathcal{I}$ is *tall* if any infinite set in $\mathbb{N}$ contains an infinite subset that belongs to $\mathcal{I}$.

(ii) $\mathcal{I}$ is *countably separated* if there is a countable family $\{X_n : n \in \mathbb{N}\}$ of subsets of $\mathbb{N}$ such that for any $A \in \mathcal{I}$ and $B \notin \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \cap X_n = \emptyset$ and $B \cap X_n \notin \mathcal{I}$.

(iii) $\mathcal{I}$ is *weakly selective* if for any partition $\{X_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $X_i \in \mathcal{I}$ for $i \geq 2$ and $\bigcup_{i\geq 2} X_i \notin \mathcal{I}$ there exists a selector of this partition which does not belong to $\mathcal{I}$.

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Let $\mathcal{I}$ be an ideal on $\mathbb{N}$.

(i) *$\mathcal{I}$ is tall if any infinite set in $\mathbb{N}$ contains an infinite subset that belongs to $\mathcal{I}$.*

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Topological representations

Theorem (M. Kwela, A. Nowik)

Ideals $\mathcal{I}_F$, $\mathcal{I}_G$ and $\mathcal{I}_K$ are tall, $F_{\sigma\delta}$ and weakly selective.

Conjecture (M. Sabok, J. Zapletal)

An ideal on a countable set has a topological representation if and only if it is tall, $F_{\sigma\delta}$ and weakly selective.

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Theorem (A. Kwela, M. Sabok)

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Thank you!

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